2013年第九屆 "IMC國際數學競賽" (新加坡)
Ninth IMC International Mathematics Contest (Singapore),2013

Answer Key to Primary Three Contest Problem

No.	1	2	3	4	5	6
Answer	D	D	C	B	A	D
No.	**7**	**8**	**9**	**10**	**11**	**12**
Answer	D	C	68	P	60	12
No.	**13**	**14**	**15**	**16**	**17**	**18**
Answer	30	21	36	1230	16	96

17. The answer is 16.

Solution:

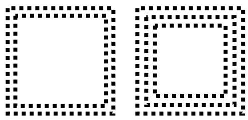
The number of delegates stayed at quadruple room is $2\times3=6$ times the number of delegates stayed of twin room,
then the number of delegates stayed at triple room is $6\div2=3$ times the number of delegates stayed of twin room.
Hence, the number of delegates stayed at twin room is

$$160\div(6+3+1)=16 \text{ delegates.}$$

Thus, number of triple room is $16\times3\div16$.

18. The answer is 96.

Solution:



We can easily trace out that the difference between the number of chess pieces on the outer layer and the inner layer of any two layers of an empty squares is 8 pieces.
Let us assume that the number of chess pieces in the most inner layer of a two layers empty square is $4a$ while the number of chess pieces in the most inner layer of a three layers empty square is $4b$.
Then $4a+(4a+8)=4b+(4b+8)+(4b+16)$,
this implies $8a+8=12b+24$, where $a\le11$, $b\le6$.
Hence, the value of a and b is at most 11 and 6; respectively.
Thus, the largest possible number of chess pieces is $11\times8+8=96$ pieces.

Answer Key to Primary Four Contest Problems

No.	**1**	**2**	**3**	**4**	**5**	**6**
Answer	D	C	D	B	D	C
No.	**7**	**8**	**9**	**10**	**11**	**12**
Answer	A	C	1053	10	26	90
No.	**13**	**14**	**15**	**16**	**17**	**18**
Answer	310	42	96	456	5.25	3809

17. The answer is 5.25.

Solution:

Let the entire journey be 42 km. Robert must spent 10 hours for travelling from town A to town B, one portion of the trip must pass thru a hill (going up a hill) and a flat road.

Time spent on the flat road is $(42-10\times3)\div(5-3)=6$ hours, while the distance for the entire flat road is $6\times5=30$ km.

Time spent on going up the hill is $10-6=4$ hours, and the distance of the hill is $4\times3=12$ km.

Time spent on going down the hill is $12\div6=2$ hours.

Therefore, the average of the speed on uphill and downhill is $42\div(6+2)=5.25$ km/h.

18. The answer is 3 809.

Solution:

The covered eight grids □□□□2013 was derived from □□□□0123,

We know that the eight grids original was 1229/123**0/123**1, where the first digit was 9 which is the units digit of 4-digit number 1229, whose position was located on $10+90\times2+900\times3+230\times4=3810^{th}$ grid.

Thus, we must move $3810-1=3809$ grids.

Answer Key to Primary Five Contest Problem

No.	1	2	3	4	5	6
Answer	C	C	B	B	C	D
No.	7	8	9	10	11	12
Answer	A	C	$\frac{35}{36}$	16	37	873
No.	13	14	15	16	17	18
Answer	4	36	50	36	4285824	(a) 7 times (b) no

17. The answer is 4 285 824.

Solution:

(a) The 4^{th} digit or D may be 5 or 0

(b) Since $8|\overline{CBA}$, this implies that $8|\overline{BA}$, but $3|\overline{AB}$, so we have $3|\overline{BA}$, that is; $24|\overline{BA}$. Hence, $\overline{BA} = 24,\ 48,\ 72,\ 96.$

(c) Since $3|\overline{CDC}$ and C is an even number, then $C = 2$ or 8.

(d) When $C = 2$, the 7-digit number is 8 425 248 (the first four digits as a 4-digit number is not multiple of 4) or 6 925 296 (the first six digits as a 6-digit number is not multiple of 7).

When $C = 4$ 285 824 or 2 785 872 (the first three digits as a 3-digit number is not multiple of 4) or 6 985 896 (the first three digits as a 3-digit number is multiple of 4).

Similarly, when $D = 0$, after checking, the number establish didn't meet the given condition

Thus, the 7-digit number is 4 285 824.

18. Answer: (a) 7 times, (b) not possible

Solution:

(a) Each time when the number of a certain term must be transform, then the position of the term will either increased or decreased by 3 or 1. (that is the same as ± 3 or ± 1).

The number located at the 20^{th} term will transform to the 1^{st} term, we must repeated skip 3 terms at a time in decreasing order.

The detail transformation is as follow:

$20 \Rightarrow 17 \Rightarrow 14 \Rightarrow 11 \Rightarrow 8 \Rightarrow 5 \Rightarrow 4 \Rightarrow 1$, a total of 7 transformations is needed so the number 20 will be in the position of the first term.

(b) In order the number in the position of 20^{th} term will be transform to the position of 1^{st} term, we must skip 19 terms of decreasing order, so we need an odd number times of transformations.

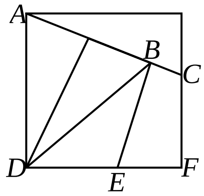
So that the number in the position of 13^{th} term will be transform to the position of the 2^{nd} term, we must skip 11 terms of decreasing order, we also need an odd number of transformation.

While the number in the position of 1^{st} term will be transform to the position of 3^{rd} term, we need to skip 2 terms of increasing order, so we need an even number times of transformation. But it is impossible!

Answer Key to Primary Six Contest Problem

No.	1	2	3	4	5	6
Answer	C	A	B	C	C	B
No.	**7**	**8**	**9**	**10**	**11**	**12**
Answer	B	B	$10\frac{7}{40}$	240	108360	0.3
No.	**13**	**14**	**15**	**16**	**17**	**18**
Answer	36	279	686	14	(a) 4:1 (b) 10:7	(a) 7 (b) 9

17. Answer: (a) 4 : 1, (b) 10 : 7



Solution:

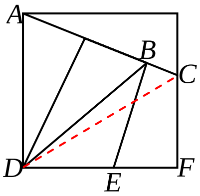
(a) Connect $\overline{CD}$, then $AB:BC=S_{\Delta ABD}:S_{\Delta BCD}=\frac{2}{5}:\left(\frac{1}{2}-\frac{2}{5}\right)=4:1.$

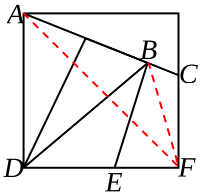
(b) Connect $\overline{FA}$ and $\overline{FB}$,

then $S_{\Delta BCF}=S_{\Delta FAB}\div 5=\left(\frac{1}{2}-\frac{1}{5}\right)\div 5=\frac{3}{50}$,

and $S_{\Delta BEF}=\frac{1}{5}-\frac{3}{50}=\frac{7}{70}$,

Thus, $DE:EF=S_{\Delta BDE}:S_{\Delta BEF}=\frac{1}{5}:\frac{7}{50}=10:7.$





18. Answer: (a) 7, (b) 9

Solution:

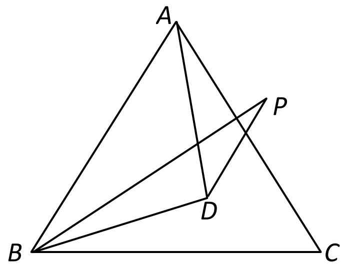
(a) Each time when the number of a certain term must be transform, then the position of the term will either increased or decreased by 3 or 1. (that is the same as ± 3 or ± 1).
The number located at the 20th term will transform to the 1st term, we must at most repeated skip 3 terms at a time in decreasing order.
The detail transformation is as follow:
$20\Rightarrow 17\Rightarrow 14\Rightarrow 11\Rightarrow 8\Rightarrow 5\Rightarrow 4\Rightarrow 1$, a total of 7 transformations is needed so the number 20 will be in the position of the first term.

(b) In order the number in the position of 20th term will be transform to the position of 1st term, we must skip 19 terms of decreasing order, so we need an odd number times of transformations.
So that the number in the position of 13th term will be transform to the position of the 2nd term, we must skip 11 terms of decreasing order, we also need an odd number of transformation.
When doing transformation each time, we can at most operate two numbers by skip $3+1=4$ position of decreasing order for at least 9 times.
Apply the transformation two times so that the number in the 20th term will move down to 14th term, then using transformation and interchange the 13th term and 20th term.
Lastly, do the transformation 6 times so that the number in the 20th term, 13th term will become the 1st and 2nd term. We need at least 9 times of transformation.

Answer Key to Junior High 1st Year Contest Problem

No.	1	2	3	4	5	6
Answer	C	B	C	B	A	D
No.	**7**	**8**	**9**	**10**	**11**	**12**
Answer	B	A	16	–11	28	$x>-3$
No.	**13**	**14**	**15**	**16**	**17**	**18**
Answer	16	0 or 4	6.5	15	30°	61

17. The Answer is 30°.



Solution:

Connect $\overline{DC}$. Then $\Delta ADC \cong \Delta BDC \cong \Delta BDP$,

$\angle P = \angle DCB = 60^{\circ} \div 2 = 30^{\circ}$.

18. The answer is 61.

Solution:

From the given information, we have

$$159 = x_1 + x_2 + \cdots + x_7$$

$$\ge x_1 + x_2 + x_3 + \frac{(x_1+3)+(x_2+2)+(x_3+1)}{3} + \frac{(x_1+4)+(x_2+3)+(x_3+2)}{3}$$

$$+\frac{(x_1+5)+(x_2+6)+(x_3+7)}{3} + \frac{(x_1+6)+(x_2+7)+(x_3+8)}{3}.$$

This implies $x_1 + x_2 + x_3 \le 62\frac{1}{7}$,

When $x_1 + x_2 + x_3 = 62$, there are no series meet the given conditions.

When $x_1 = 19, x_2 = 20, x_3 = 22, x_4 = 23, x_5 = 24, x_6 = 25, x_7 = 26$, such that the largest possible value of $x_1 + x_2 + x_3 = 19 + 20 + 22 = 61$.

Answer Key to Junior High 2nd Year Contest Problem

No.	1	2	3	4	5	6
Answer	B	D	D	C	D	B
No.	**7**	**8**	**9**	**10**	**11**	**12**
Answer	C	D	$y=\frac{1}{3}x+1$	1 or −2	$-\frac{671}{15}$	45
No.	**13**	**14**	**15**	**16**	**17**	**18**
Answer	−1	8	$\frac{1}{10000}$	1	$\frac{37}{56}$	（见详解）

17. The answer is $\frac{37}{56}$.

Solution:

(a) The number of combination that Mary and Nestor select the three-digit number is $C_9^3 \times C_8^3$.

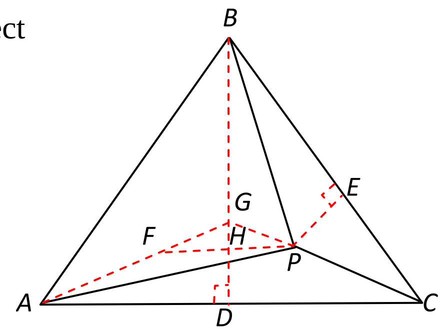
(b) Case 1: Mary select the digit 9, and the number of ways that Nestor can select is $C_8^2 \times C_8^3$.

Case 2: Mary select a digit other than 9, then the number of combinations that each of them, that is Mary or Nestor select the three-digit numbers (whether equal or unequal) is $\frac{C_8^3 \times C_8^3 - C_8^3}{2}$.

Therefore, the total probability is

$$\frac{C_8^2 \times C_8^3 + \frac{C_8^3 \times C_8^3 - C_8^3}{2}}{C_9^3 \times C_8^3} = \frac{C_8^2 + \frac{C_8^3 - 1}{2}}{C_9^3} = \frac{28 + \frac{55}{2}}{84} = \frac{37}{56}.$$

18. Proof:



(a) Construct the line of symmetry $\overline{BD}$ of ΔABC and intersect the extension of $\overline{CP}$ at point G.

Connect $\overline{AG}$, through P construct $\overline{PF} \perp \overline{BD}$ intersect $\overline{AG}$ and $\overline{BD}$ at F and H; respectively.

(b) From Properties of Symmetry, $\angle GAC = \angle GCA = 20^\circ$, then $\overline{AP}$ is the angle bisector of $\angle GAC$, Since $\overline{PF} \parallel \overline{AC}$, then $\angle FPA = \angle CAP = \angle FAP$,

Hence, $FP = FA$.

(c) Using the Property of Symmetry again, we have $FP = FA = PC$.

Thus, $PH = \frac{1}{2}PF = \frac{1}{2}PC$.

(d) Through point P construct $\overline{PE} \perp \overline{BC}$ at point E.

Since in $\mathrm{Rt}\Delta PEC$, $\angle PCB = 30^\circ$,

then $PE = \frac{1}{2}PC$.

Therefore, $PE = PH$, Thus, $\overline{BP}$ bisects $\angle CBD$.

(e) Since $\overline{BD}$ bisects $\angle ABC$,

Therefore, $\angle ABP = \angle ABD + \angle PBD = \frac{1}{2}\times 80^\circ + \frac{1}{4}\times 80^\circ = 60^\circ$.

Answer Key to Junior High 3rd Year Contest Problem

No.	1	2	3	4	5	6	7	8
Answer	B	D	B	C	D	C	C	A
No.	9	10	11	12	13	14	15	16
Answer	-1	73、649657、92737	$\frac{5\sqrt{2}}{2}$	45	$-\frac{671}{15}$	1	$\frac{1}{110}$	80°

17. Answer: 36°, 108°, 36°

Solution: Let the three angles of ΔABC as $\angle A, \angle B$ and $\angle C$.

The size of each angle bisector is $\angle BPC = \dfrac{\angle BDC + \angle BEC}{2} = 45^\circ + \dfrac{3}{4}\angle A.$

(a) If $\angle A$ is an acute angle, then by Exterior Angle Theorem of a Triangle, we know that

$\angle BOC = 2\angle A.$ But $BPOC$ is a cyclic quadrilateral, then $\angle BPC = \angle BOC,$

this implies $\angle A = 36^\circ.$ $\Rightarrow$ (4 points)

Since $PB = PC,$ this implies P is the center of arc BO.

So, $\angle BCP = \dfrac{3}{4}\angle C = \dfrac{1}{2}\angle BCO = 27^\circ$, then $\angle C = 36^\circ, \angle B = 108^\circ.$ $\Rightarrow$ (4 points)

(b) If $\angle A$ is a right angle or obtuse angle, then by Exterior Angle Theorem of a Triangle,

we have $\angle BOC = 360^\circ - 2\angle A.$

Four points B, P, O and C will lie on a circle if and only if $\angle BPC + \angle BOC = 180^\circ,$

so $\angle A = 180^\circ,$ which didn't meet the given condition of the problem. $\Rightarrow$ (2 points)

18. Answer: (a) no (b) no (c) 1935 or 2115

(a) The parity of $A+B+C$ under the Transformation #1 and #2 remain unchanged,

and $1+0+(-1)=0$ is an even number, so it is impossible $2012+2013+2014$

to be an odd number.

Thus, it is not possible! $\Rightarrow$ (3 points)

(b) The value of $A+B+C \pmod 4$ under the Transformation #1 and #2 remain unchanged or it will be $-(A+B+C)(\bmod 4)$; so

$1+0+(-1)\equiv 0(\bmod 4)$, impossible to obtain $2013+2014+2015\equiv 2(\bmod 4)$.

Thus, the answer is NO! $\Rightarrow$ (4 points)

(c) Hence, we can construct the following structure:

$$(-1,0,1)\to(0,1,3)\to(1,3,8)\to(1,8,15)\to(1,15,24)\to\cdots$$

$$\to\left(0,44^2-1=1935,45^2-1=2024\right)\to\left(0,45^2-1,46^2-1=2115\right).$$

Therefore, the value of x is either 1935 or 2115.

This implies to obtain all the possible values of x may need to construct a 2nd degree equation by Vieta's Theroem, that is; the value of

$A^2+B^2+C^2-2AB-2BC-CA$ remain unchanged even under several transformation $\Rightarrow$ (3 points)

Answer Key to Senior High 1st Year Contest Problem

No.	1	2	3	4	5	6	7	8
Answer	C	A	C	A	C	A	C	A
No.	9	10	11	12	13	14	15	16
Answer	$\frac{17}{4}$	4008	$-\frac{\sqrt{2}}{2}$	$(-\infty,-1]\cup[1,+\infty)$	73、649657、92737	2	$\frac{1}{110}$	1

17. The Answer is 756.

Solution: Sequentially put the numbers 1, 4, 7, 2, 5, 8, 3, 6 to the eight vertices of an octagon such that the difference between two numbers is 3 or 4 or 5 if and only if these two numbers are neighborhood or the two end points of the main diagonal. We can easily prove that: For any eight consecutive numbers, we can select at most 3 numbers such that their difference is not equal to 3, 4 or 5. Hence, from 1 to 2013, we may select at most $3\times 252=756$ numbers to meet our requirements.$\Rightarrow$ (5 points)

We can also construct a structure to obtain the discussion as described above: We may select any or all of the following format: $8k+1,\ 8k+2,\ 8k+3.\Rightarrow$ (5 points)

18. Answer: (a) no (b) no (c) 1935 or 2115

(a) The parity of $A+B+C$ under the Transformations #1 and #2 remain unchanged,

and $1+0+(-1)=0$ is an even number, so it is impossible $2012+2013+2014$

to be an odd number.

Thus, it is not possible! $\Rightarrow$ (3 points)

(b) The value of $A+B+C\pmod 4$ under the Transformations #1 and #2 remain

unchanged or it will be $-(A+B+C)\pmod 4$; so

$1+0+(-1)\equiv 0\pmod 4$, It is impossible to obtain

$2013+2014+2015\equiv 2\pmod 4$.

Thus, the answer is NO! $\Rightarrow$ (4 points)

(c) Hence, we can construct the following structure:

$$(-1,0,1)\to(0,1,3)\to(1,3,8)\to(1,8,15)\to(1,15,24)\to\cdots$$

$$\rightarrow\left(0, 44^2-1=1935, 45^2-1=2024\right) \rightarrow\left(0, 45^2-1, 46^2-1=2115\right).$$

Therefore, the value of x is either 1935 or 2115.

This implies, to obtain all the possible values of x may require to construct a 2^{nd} degree equation by Vieta's Theroem, that is; the value of

$A^2+B^2+C^2-2AB-2BC-CA$ remains unchanged even under several transformations $\Rightarrow$ (3 points)

Answer Key to Senior High 2nd Year Contest Problem

No.	1	2	3	4	5	6	7	8
Answer	B	A	C	D	C	A	C	A
No.	9	10	11	12	13	14	15	16
Answer	$(-3,\frac{1-\sqrt{5}}{2})\cup(\frac{-1+\sqrt{5}}{2},3)$	$\frac{3\sqrt{3}}{4}$	100	2	73、649657、92737	1	$\frac{1}{110}$	1

17. The answer is 915.

Solution: Sequentially put the numbers 1, 6, 11, 5, 10, 4, 9, 3, 8, 2, 7 to 11 vertices of a regular 11-side polygon such that the difference between two numbers is 5 or 6 if and only if these two numbers are neighborhood. We can easily prove that: For any eleven consecutive numbers, we can select at most 5 numbers such that their difference of any two numbers is not equal to 5 or 6. Hence, from 1 to 2013, we may select at most $5\times 183=915$ numbers to meet our requirements. $\Rightarrow$ (5 points)

We can also construct a structure to obtain the discussion as described above: We may select any or all of the following format: $11k+1$, $11k+11$, $11k+10$, $11k+9$, $11k+8$.

$\Rightarrow$ (5 points)

18. Answer: (a) no (b) no (c) 1935 or 2115

(a) The parity of $A+B+C$ under the Transformations #1 and #2 remain unchanged,

and $1+0+(-1)=0$ is an even number, so it is impossible $2012+2013+2014$

to be an odd number.

Thus, it is not possible! $\Rightarrow$ (3 points)

(b) The value of $A+B+C\pmod 4$ under the Transformations #1 and #2 remain

unchanged or it will be $-(A+B+C)\pmod 4$; so

$1+0+(-1)\equiv 0\pmod 4$, It is impossible to obtain

$2013+2014+2015\equiv 2\pmod 4$.

Thus, the answer is NO! $\Rightarrow$ (4 points)

(c) Hence, we can construct the following structure:

$$(-1,0,1) \to (0,1,3) \to (1,3,8) \to (1,8,15) \to (1,15,24) \to \cdots$$

$$\to \left(0, 44^2 - 1 = 1935, 45^2 - 1 = 2024\right) \to \left(0, 45^2 - 1, 46^2 - 1 = 2115\right).$$

Therefore, the value of x is either 1935 or 2115.

This implies, to obtain all the possible values of x may require to construct a 2nd degree equation by Vieta's Theroem, that is; the value of

$A^2 + B^2 + C^2 - 2AB - 2BC - CA$ remains unchanged even under several transformations $\Rightarrow$ (3 points)